hep-th/9704082

Global Conformal Invariance in D-dimensions and Logarithmic Correlation Functions

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Abstract

We define transformation of multiplets of fields (Jordan cells) under the D-dimensional conformal group, and calculate two and three point functions of fields, which show logarithmic behaviour. We also show how by a formal differentiation procedure, one can obtain n-point function of logarithmic field theory from those of ordinary conformal field theory.

PACS: 11.25.Hf

Keyword: Conformal Field Theory

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1 Introduction

Recently, Logarithmic Conformal Field Theories (LCFT) have been studied in a series of papers [1]-[13], both for their pure theoretical interest concerning the structure and classification of conformal field theories [1, 2, 3, 4, 5] and for their relevance in some physical systems [6, 7, 8, 9, 10, 11] which are described by non-unitary or non-minimal conformal models.

All such works have dealt with two dimensional conformal field theory [1] relying heavily on the underlying Virasoro algebra, and have described how the appearance of logarithmic singularities is related to the modification of the representation of the Virasoro algebra.

In this paper we will try to understand LCFT's from yet another point of view, that's we consider d-dimensional conformal invariance.

Although our results are based on a generalization of the basic idea of [1] , we hope that by changing or simplifying the context of study, we can add a little bit to the understanding of the subject in general.

As is well known, one of the basic assumptions of conformal field theory is the existence of a family of operators, called scaling fields, which transform under scaling $S : x \to x' = \lambda x$, simply as follows:

$$\phi(x) \to \phi'(x') = \lambda^{-\Delta}\phi(x), \tag{1.1}$$

where Δ is the scaling weight of $\phi(x)$. It is also assumed that under the conformal group, such fields transform as,

$$\phi(x) \to \phi'(x') = \parallel \frac{\partial x'}{\partial x} \parallel^{\frac{-\Delta}{d}} \phi(x), \tag{1.2}$$

where d is the dimension of space and $\parallel \frac{\partial x'}{\partial x} \parallel$ is the Jacobian of the transformation. Equation (1.2) which encompasses eq. (1.1) defines the transformation of the quasi-primary fields. For future use we note that the Jacobian equals λ^d for scaling transformation and $\parallel x \parallel^{-2d}$ for the Inversion transformation $I : x \to x' = \frac{x}{\|x\|^2}$, being unity for the other elements of the conformal group. Combination of (1.2) with the definition of symmetry of the correlation functions, i.e.:

$$< \phi_1'(x_1') \cdots \phi_N'(x_N') >=< \phi_1(x_1') \cdots \phi_N(x_N') >, \tag{1.3}$$

allows one to determine the two and the three point functions up to a constant and the four point function up to a function of the cross ratio.

It's precisely the assumption that scaling fields constitute irreducible representations of the scaling transformation, which imposes power law singularity on the correlation functions. As we will see, relaxing this assumption, one naturally arrives at logarithmic singularities. It also leads to many other pecularities, in the relation between correlation functions. To begin with, we consider a multiplet of fields,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}, \tag{1.4}$$

and note that under scaling $x \to \lambda x$, the most general form of the transformation of Φ is,

$$\Phi(x) \to \Phi'(x') = \lambda^{T'} \Phi(x) \tag{1.5}$$

where T' is an arbitrary matrix. More generally, we replace (1.5) by,

$$\Phi(x) \to \Phi'(x') = \parallel \frac{\partial x'}{\partial x} \parallel^{T} \Phi(x). \tag{1.6}$$

where T is an $n \times n$ arbitrary matrix. When T is diagonalizable, one arrives at ordinary scaling fields by redefining Φ, so that all the fields transform as 1-dimensional representation. Otherwise, following [12] we assume that T has Jordan form,

$$T = \begin{pmatrix} \frac{-\Delta}{d} & 0 & \cdots & 0 \\ 1 & \frac{-\Delta}{d} & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & \cdots & 1 & \frac{-\Delta}{d} \end{pmatrix}. \tag{1.7}$$

Rewriting T as $\frac{-\Delta}{d}1 + J$, where $J_{ij} = \delta_{i-1,j}$, eq. (1.6) can be written in the form,

$$\Phi(x) \to \Phi'(x') = \parallel \frac{\partial x'}{\partial x} \parallel^{\frac{-\Delta}{d}} \Lambda_x \Phi(x), \tag{1.8}$$

where $\Lambda_x = \parallel \frac{\partial x'}{\partial x} \parallel^{J}$ is a lower triangular matrix of the form,

$$(\Lambda_x)_{ij} = \frac{\{\ln \parallel \frac{\partial x'}{\partial x} \parallel\}^{i-j}}{(i-j)!}, \qquad (\Lambda_x)_{ii} = 1, \tag{1.9}$$

i. e. for $N = 2$ we have,

$$\begin{aligned} \phi_1'(x') &= \parallel \frac{\partial x'}{\partial x} \parallel^{\frac{-\Delta}{d}} \phi_1(x), \\ \phi_2'(x') &= \parallel \frac{\partial x'}{\partial x} \parallel^{\frac{-\Delta}{d}} \big(\ln \parallel \frac{\partial x'}{\partial x} \parallel \phi_1(x) + \phi_2(x)\big). \end{aligned} \tag{1.10}$$

An important point is that the top field $\phi_1(x)$ *always* transform as an ordinary quasi-primary field. A most curious propety of the transformation (1.8) is that each field ϕ_{k+1} transforms as if it is a formal derivative of ϕ_k with respect to $\frac{-\Delta}{d}$,

$$\phi_{k+1}(x) = \frac{1}{k}\frac{\partial}{\partial(\frac{-\Delta}{d})}\phi_k(x). \tag{1.11}$$

This formal relation which determines the transformation of all the fields of a Jordan cell from that of the top field ϕ_1, essentially means that with due care, one can determine the correlation functions of the lower fields from those of the ordinary top fields simply by formal differentiation. We will elucidate this point later on. The phrase with due care in the previous statement refers to the two point function of fields within *the same* Jordan cell.

2 The Two Point Functions

We already know by standard arguments [13] that the two point function of the top fields ϕ_α and ϕ_β belonging to two different Jordan cells (Δ_α, n) and (Δ_β, m) vanishes, i.e.:

$$< \phi_\alpha(x)\phi_\beta(y) >= \frac{A\delta_{\Delta_\alpha,\Delta_\beta}}{\parallel x-y \parallel^{2\Delta_\alpha}}. \tag{2.1}$$

Due to the observation (1.11), it follows that the two point function of all the fields of two different Jordan cells with respect to each other vanish. Therefore in this section we calculate the two point function of the fields within the same Jordan cell. As we will see logarithmic conformal symmetry gives many interesting and unexcepted results in this case. Let's denote the matrix of two point functions $< \phi_i(x)\phi_j(y) >$ for all $\phi_i, \phi_j \in (\Delta, n)$ by $G(x, y)$, then from rotation and translation symmetries, this matrix should depends only on $\parallel x - y \parallel$. From scaling symmetry and using (1.2) and (1.3), we will have,

$$\Lambda G(\parallel x-y \parallel)\Lambda^t = \lambda^{2\Delta} G(\lambda \parallel x-y \parallel), \tag{2.2}$$

where $\Lambda = \lambda^{dJ}$, and from inversion symmetry, we have,

$$\Lambda_x G(\parallel x-y \parallel)\Lambda_y^t = \parallel x-y \parallel^{-2\Delta_\alpha} G(\frac{\parallel x-y \parallel}{\parallel x \parallel \parallel y \parallel}), \tag{2.3}$$

where

$$\Lambda_x = \parallel x \parallel^{-2dJ},$$

$$\Lambda_y \quad = \quad \| y \|^{-2dJ} . \tag{2.4}$$

Defining the matrix F as $G(\| x - y \|) = \frac{F(\|x-y\|)}{\|x-y\|^{2\Delta}}$, we will have from (2.2) and (2.3),

$$\Lambda F(\| x - y \|)\Lambda^t = F(\lambda \| x - y \|), \tag{2.5}$$

and

$$\Lambda_x F(\| x - y \|)\Lambda_y^t = F(\frac{\| x - y \|}{\| x \|\| y \|}). \tag{2.6}$$

For every arbitrary λ, we now choose the points x and y such that $\| x \|= \lambda^{\frac{-1}{4}}$ and $\| y \|= \lambda^{\frac{-3}{4}}$. *It should be noted that in this way by varying λ, we can span all the points of space.* From (2.4), we will have $\Lambda_x = \Lambda^{\frac{1}{2}}$ and $\Lambda_y = \Lambda^{\frac{3}{2}}$, therefore eq. (2.5) turns into,

$$\Lambda^{\frac{1}{2}} F(\| x - y \|)\left(\Lambda^{\frac{3}{2}}\right)^t = F(\lambda \| x - y \|). \tag{2.7}$$

Combining (2.5) and (2.7) and using invertibility of Λ, we arrive at,

$$F = \Lambda^{\frac{1}{2}} F\left(\Lambda^t\right)^{\frac{-1}{2}}, \tag{2.8}$$

by iterating (2.8), we will have $F = \Lambda F \Lambda^t$, and by rearranging, we will have,

$$F\Lambda^t = \Lambda F. \tag{2.9}$$

Expanding Λ in terms of power of $\ln\lambda$ as $\Lambda = 1 + (d\ln\lambda)J + \frac{(d\ln\lambda)^2}{2!}J^2 + \cdots$ and comparing both sides, we arrive at

$$F\left(J^t\right)^k = \left(J\right)^k F, \qquad k = 1, 2, \cdots, n-1. \tag{2.10}$$

Since $\left(J^k\right)_{ij} = \delta_{i,j+k}$, we will have from (2.10),

$$F_{i,j-k} = F_{i-k,j}, \tag{2.11}$$

which means that on each opposite diagonal of the matrix F, all the correlations are equal. Moreover from $FJ = JF$, one obtains,

$$\sum_{l=1}^{n} F_{il}\delta_{j,l+1} = \sum_{l=1}^{n} \delta_{i,l+1}F_{lj}, \tag{2.12}$$

which means that if $j = 1$ and $1 < i \leq n$, then $F_{i-1,j} = 0$, or

$$F_{ij} = 0 \qquad \text{for} \qquad j = 1 \qquad \text{and} \qquad 1 \leq i \leq n-1. \tag{2.13}$$

Combining this with (2.11), we find that all the correlations above the opposite diagonal are zero. In order to find the final form of F, we use eq. (2.5) again, this time in infinitesimal form, let $\Lambda = 1 + \alpha J + o(\alpha^2)$ where $\alpha = d \ln \lambda$, then from $\Lambda F(x) \Lambda^t = F(\lambda x)$, we have,

$$d\big(JF + FJ^t\big) = x \frac{dF}{dx}. \tag{2.14}$$

Due to the property (2.11) only the last column of F should be found, therefore from (2.14) we obtain,

$$\begin{aligned} x\frac{dF_{1,n}}{dx} &= dF_{1,n-1} \equiv 0, \\ x\frac{dF_{i,n}}{dx} &= 2dF_{i,n-1}, \qquad \text{if} \quad i > 1 \end{aligned} \tag{2.15}$$

which upon introducing the new variable $y = 2d \ln x$ gives,

$$F_{1,n} = c_1, \quad F_{2,n} = c_1 y + c_2, \quad F_{3,n} = \frac{1}{2} c_1 y^2 + c_2 y + c_3, \quad \text{etc.} \tag{2.16}$$

with the recursion relations,

$$\frac{dF_{i,n}}{dy} = F_{i-1,n}. \tag{2.17}$$

Thus we have arrived at the final form of the matrix F, which is as follows:

$$F = \begin{pmatrix} 0 & \cdots & 0 & 0 & g_0 \\ 0 & \cdots & 0 & g_0 & g_1 \\ 0 & \cdots & g_0 & g_1 & g_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_0 & \cdots & g_{n-2} & g_{n-1} & g_n \end{pmatrix}, \tag{2.18}$$

where each g_i is a polynomial of degree i in y, and $g_i = \frac{dg_{i+1}}{dy}$. All the correlations depend on the n constants $c_1, \cdots, c_n$, which remain undetermined. We have checked (as the reader can check for the single $n = 2$ case) that inversion symmetry puts no further restrictions on the constants c_i.

3 n-Point Correlation Functions

The observation (1.11) that the transformation properties of the members of a Jordan cell are as if the are formal derivative of the top field in the cell, allows one to determine the correlation functions of all the fields within a single or different Jordan cells, once the correlation function of the top fields are determined. As

an example, from ordinary CFT, we know that conformal symmetry completely determines the three point function up to a constant. Let ϕ_α, ϕ_β and ϕ_γ be the top fields of three Jordan cells (Δ_α, l), (Δ_β, m) and (Δ_γ, n) respectively. Therefore we know that,

$$< \phi_\alpha(x)\phi_\beta(y)\phi_\gamma(z) >= \frac{A_{\alpha\beta\gamma}}{\| x-y \|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma} \| x-z \|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta} \| y-z \|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}}, \tag{3.1}$$

where the constant $A_{\alpha\beta\gamma}$ in principle depends on the weights $\Delta_\alpha, \Delta_\beta$ and Δ_γ. Denoting the second field of the cell (Δ_α, l) by $\phi_{\alpha 1}$,[3] we will readily find from (3.1) that,

$$\begin{aligned} < \phi_{\alpha 1}(x)\phi_\beta(y)\phi_\gamma(z) > &= -d\frac{\partial}{\partial \Delta_\alpha} < \phi_\alpha(x)\phi_\beta(y)\phi_\gamma(z) > \\ &= \frac{A'_{\alpha\beta\gamma}}{\| x-y \|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma} \| x-z \|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta} \| y-z \|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}} \\ &+ \frac{dA_{\alpha\beta\gamma}}{\| x-y \|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma} \| x-z \|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta} \| y-z \|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}} \ln\left(\frac{\| y-z \|}{(\| x-y \|)(\| x-z \|)}\right), \end{aligned} \tag{3.2}$$

where $A'_{\alpha\beta\gamma} = -d\frac{\partial}{\partial\Delta_\alpha} A_{\alpha\beta\gamma}$ is a new undetermined constant. For the correlation functions of fields within a single cell, one should then take the limit $\beta, \gamma \to \alpha$ in the above formula. It's not difficult to check that this formula satisfies all the requirements demanded by conformal symmetry. No need to say one can continue this procedure of formal differentiation to determine other correlation functions. It may be asked why we have not applied the procedure of formal differentiation for calculation of two point function, but have followed an independent route. The point is that, the two point function has the form $< \phi_\alpha(x)\phi_\beta(y) >= \frac{A\delta_{\Delta_\alpha,\Delta_\beta}}{\|x-y\|^{2\Delta_\alpha}}$, which upon differentiation, yields derivative of the Kronecker symbol or the delta function which is not easy to handle.

4 The Operator Product Expansion

In this section we calculate the operator product of the fields in two different Jordan cell, $(\Delta_\alpha, N_\alpha)$, and$(\Delta_\beta, N_\beta)$. We assume that the operator product of the two top-fields $\phi_\alpha \in (\Delta_\alpha, N_\alpha)$ and $\phi_\beta \in (\Delta_\beta, N_\beta)$ can be expressed in terms of the fields and their descendants in other Jordan cells,

$$\phi_\alpha(x)\phi_\beta(y) = \sum_\gamma \sum_{n=1}^{N_\gamma} C^\gamma_{\alpha\beta,n}(\| x-y \|)\phi_{\gamma,n}(y), \tag{4.1}$$

[3]For simplicity, we have denoted the top field by ϕ and the second field by ϕ_1, instead of ϕ_1 and ϕ_2 respectively.

where the sum is over the Jordan cells of weight $(\Delta_\gamma, N_\gamma)$. The field $\phi_{\gamma,n}$ stand for the $n-$th element of a Jordan cell together with it's descendants. We now demand that both sides have the same behaviour under conformal transformations, i. e. we demand that (4.1) is equivalent to the following relation,

$$\phi'_\alpha(x')\phi'_\beta(y') = \sum_{n,\gamma} C^\gamma_{\alpha\beta,n}(\| x' - y' \|)\phi'_{\gamma,n}(y'). \tag{4.2}$$

Considering first the scale transformation, we obtain from (1.6),

$$\lambda^{-\Delta_\alpha - \Delta_\beta}\phi_\alpha(x)\phi_\beta(y) = \sum_{n,\gamma,p\leq N_\gamma} C^\gamma_{\alpha\beta,n}(\lambda(\| x - y \|))\lambda^{-\Delta_\gamma}\Lambda_{np}\phi_{\gamma,p}(y). \tag{4.3}$$

Redefining the function $C^\gamma_{\alpha\beta,n}(x)$ as follows,

$$C^\gamma_{\alpha\beta,n}(x) = \frac{C^\gamma_{\alpha\beta} f_n(x)}{\| x - y \|^{\Delta_\alpha + \Delta_\beta - \Delta_\gamma}}, \tag{4.4}$$

where $C^\gamma_{\alpha\beta}$'s are constant and comparing eqs. (4.1) with (4.3) leads to the following equation for f's,

$$f_n(\lambda^{-1}x) = \sum_{p=1}^{N_\gamma} f_p(x)\Lambda_{pn}, \tag{4.5}$$

or in compact matrix form,

$$f(\lambda^{-1}x) = \Lambda^T f(x), \tag{4.6}$$

Equation (4.6) is easily solved. It is in fact a recursion relation for f_n's due to the triangular form of Λ. The general solution is,

$$f_i(x) = \sum_{k=0}^{N-i} \frac{\alpha_{i+k}(-d\ln x)^k}{k}, \tag{4.7}$$

where the constants $\alpha_1, \cdots, \alpha_N$ are free. For $N = 3$ for example, we have,

$$\begin{aligned}
f_3(x) &= \alpha_3 \\
f_2(x) &= \alpha_3(-d\ln x) + \alpha_2 \\
f_1(x) &= \frac{1}{2}\alpha_3(-d\ln x)^2 + \alpha_2(-d\ln x) + \alpha_1.
\end{aligned} \tag{4.8}$$

It's interesting to note that the following relation holds among these functions,

$$f_k(x) = \frac{\partial}{\partial(-d\ln x)} f_{k-1}(x). \tag{4.9}$$

5 Conjugation Properties

In this section, we will restrict ourselves to two dmensions and derive the hermitian conjugate of the fields in a Jordan cell. Our main rresult is encoded in the following formula,

$$\hat{\Phi}^{\dagger}(z) = \hat{\Phi}^{t}(\frac{1}{\bar{z}})\bar{z}^{-2(\Delta - J^{t})} \tag{5.1}$$

where $\hat{\Phi}$ is an N dimensional multiplet of operators, the superscript t, denotes transpose and the $\bar{z}$ dependence of the fields have been supressed. As an example, when $N = 2$ and $\hat{\Phi} = \begin{pmatrix} \hat{\phi} \\ \hat{\psi} \end{pmatrix}$, we have,

$$\begin{aligned} \hat{\phi}^{*}(z) &= \bar{z}^{-2\Delta}\hat{\phi}(\frac{1}{\bar{z}}), \\ \hat{\psi}^{*}(z) &= \bar{z}^{-2\Delta}(\hat{\psi}(\frac{1}{\bar{z}}) + 2\ln\bar{z}\hat{\phi}(\frac{1}{\bar{z}})). \end{aligned} \tag{5.2}$$

As a justification for the validity of (5.1), consider the transformation $z \to \frac{1}{\bar{z}}$, with $\parallel z \parallel > \parallel w \parallel$ and denote $< \phi_i(z)\phi_j(w) >$ by $< \Phi(z)\Phi^t(w) >_{ij}$, then,

$$< \Phi(z)\Phi^{t}(w) >=< 0|\hat{\Phi}(z)\hat{\Phi}^{t}(w)|0 >=< 0|R\hat{\Phi}^{\dagger t}(w)\hat{\Phi}^{t}(z)|0 >^{cc} \tag{5.3}$$

where cc denotes complex conjugation and $|0 >$ denotes the vaccum. The right hand side of (5.3) is then equal to,

$$\begin{aligned} < 0|\bar{w}^{-2(\Delta-J)}\hat{\Phi}(\frac{1}{\bar{w}})\hat{\Phi}^{t}(\frac{1}{\bar{z}})\bar{z}^{-2(\Delta-J^{t})}|0 >^{cc} &= w^{-2(\Delta-J)} < 0|\hat{\Phi}(\frac{1}{w})\hat{\Phi}^{t}(\frac{1}{z})|0 > z^{-2(\Delta-J^{t})} \\ &= w^{-2(\Delta-J)} < \Phi(\frac{1}{w})\Phi^{t}(\frac{1}{z}) > z^{-2(\Delta-J^{t})} \\ &= < \Phi'(w)\Phi'^{t}(z) >, \end{aligned} \tag{5.4}$$

which is the desired equality.

Acknowledgement

We would like to thank A. Aghamohammadi for valuable discussions.

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